

A fable story for understanding eigenvalues titled “Enlargement Factors of the Magnification Machine are Eigenvalues”¹

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I was wondering how you studied eigenvalues when you were learning linear algebra.

First, given a matrix, you would calculate the eigenvalues. Then you would find eigenvectors for the eigenvalues and finally obtain the diagonal matrix. Many of you may learn eigenvalues following this flowchart.

I was wondering, however, whether this approach would enable you to understand the concept of eigenvalues; the eigenvalues are invariant under a change of a basis matrix.

If you are a student in a mathematics department, you could have repeatedly heard about eigenvalues in your lectures. Then you would be able to grasp the concept for yourself. However, I wonder how other department students would be able to comprehend the concept. I did not think that once or twice given explanations enabled you understand the concept.

I teach mathematics in a department of administration. So our students are not good at mathematics compared with math department students. However they will have to learn the eigenvalues for multivariate data analysis. To have our students understand the eigenvalues, I contrived a means for the explanation. The key word is an enlargement factor of the magnification machine. I need a further explanation for this. Then I would be grateful if you would read the following fable story I made.

Using the teaching methods, many students have comprehended the eigenvalue concept. Because I think this method is efficient, I would like to pervade the method. So I have written the following fable story. Now I will start the story.

*** * * Enlargement Factors of the Magnification Machine are Eigenvalues * * ***

Long time ago, there was an animal kingdom. The five small countries named CENTRE, SOUTH, EAST, NORTH, and WEST are segments of the animal kingdom.

As you know well, the space in which we live is three dimensional. However, suppose that the animal kingdom is two dimensional, for simplification of the following description.

In the two dimensional world, the world form is a plane. There is no concept of the vertical height. They cannot feel the height, as we cannot feel the fourth dimension. (Please do not think of details of

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the two dimensional world. I adopted and used a two dimensional world just because of simplification of the matrix description.)

As the centre of the kingdom, there is a country named CENTRE which is also the cultural central hub. So many foreign students from other countries go to CENTRE to study innovative technologies there. Here in CENTRE, a genius dog named Mr Orthom born in CENTE and a genial cat named Ms Diagonary from WEST study together in University of CENTRE.

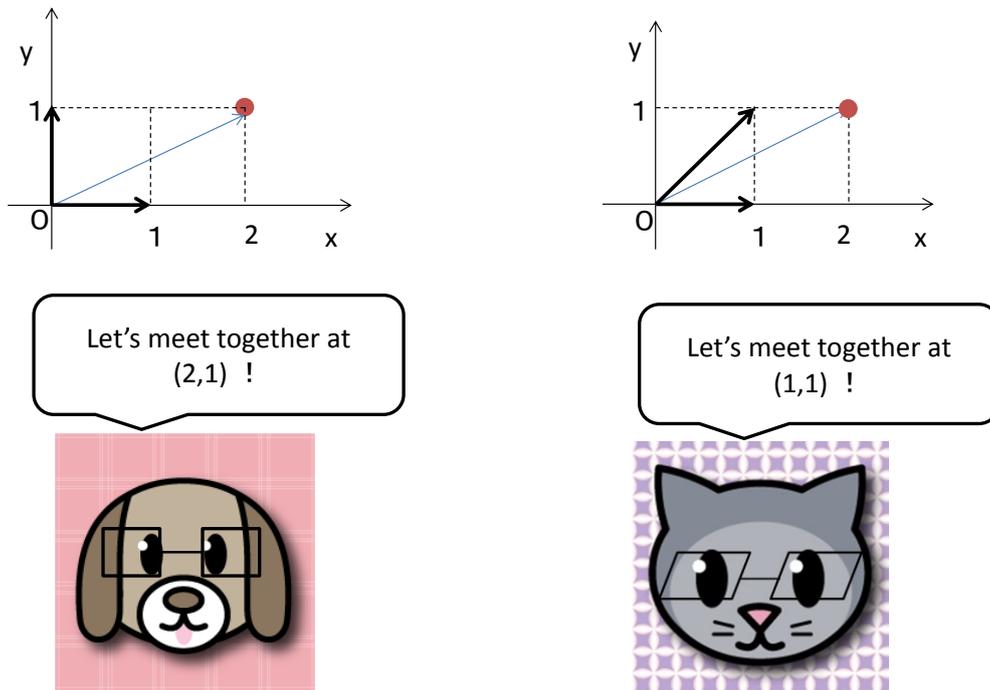
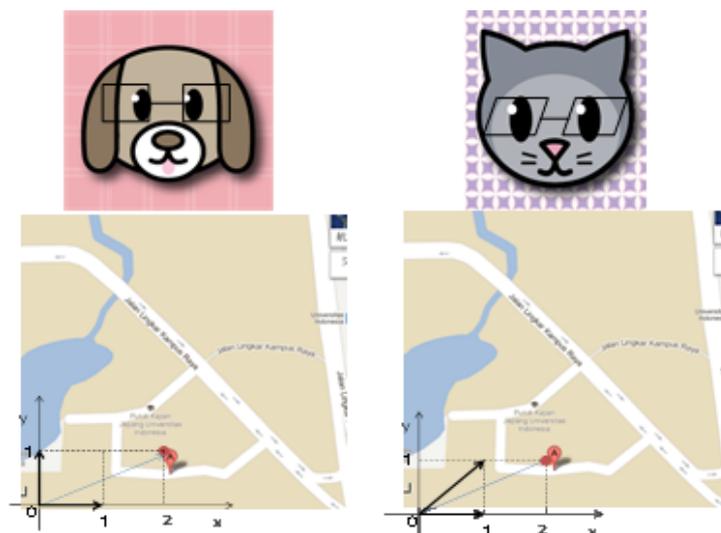


Figure 1: A basis of CENTRE is orthogonal and a basis of WEST is diagonal.



Now let me explain a bit a basis of a vector space to kindle your memory about that.

In linear algebra, a basis is a set of linearly independent vectors which define a coordinate system. Given a basis of a vector space, every element of the vector space can be expressed uniquely as a finite linear combination of basis vectors.

Let's consider a basis of CENTRE. A basis of CENTRE is an orthogonal basis: namely the two basis vectors are mutually orthogonal. On the other hand, the basis of WEST is diagonal (See Figure 1).

Figuratively speaking, Mr Orthom wears the orthogonal glasses and Ms Diagonary wears the diagonal glasses as shown there. So when they express the same point, they represent the point by different coordinates. This is because the bases they use are different. The coordinates are expressed in this world by an ordered pair (a, b) of real numbers. Although you may be anxious about their gathering, there is no trouble. Even if the coordinates are different ones, they can get together at the spot. The image of both them is the same spot as illustrated as red marks in Figure 1.

I will show you a more concrete example. Suppose that Mr Orthom said "Let's meet together at the coordinate $c = (c_1, c_2) = (2, 1)$." Then, Ms Diagonary would say "Yes, let's meet together at the coordinate $w = (w_1, w_2) = (1, 1)$ " after the change of the basis matrix as follows:

Solve the equation to find w_1 and w_2 :
$$w_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + w_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

The answer is $w_1=1, w_2=1$.

One day, at the University laboratory, Mr Orthom invented the Magnification Machine. This is the innovative great invention. Input an image, the M Machine magnifies the image.

The magnification transformation can be expressed by the following diagonal matrix:

$$\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$$

Input the vector (1, 1), the M machine outputs the vector (3, 2) as shown in Figure 2.

$$\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

In the initial experiments, the machine output the sheared (skewed) images. The shearing transformation is expressed for example as follows:

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

After many improvements, Mr Orthom accomplished the magnifying machine of which enlargement factors can be defined independently for each basis vectors. The matrix is expressed as a diagonal matrix.

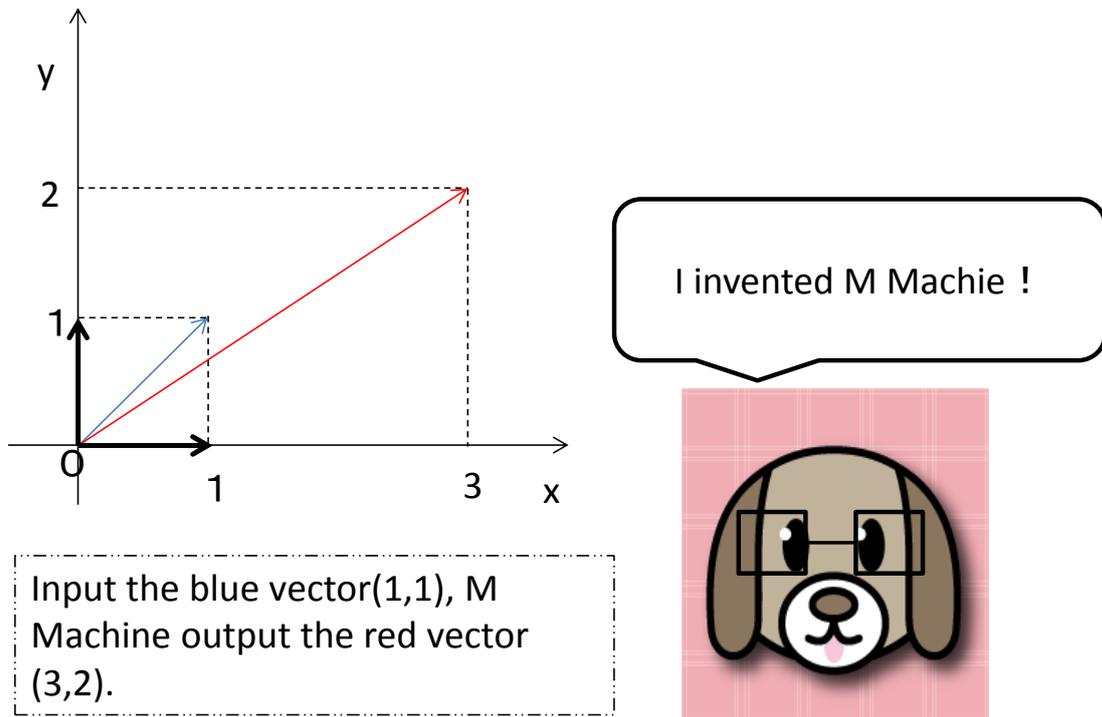


Figure 2: The M Machine magnifies an image. The image transformation is neither a shearing nor a reflection. The enlargement factors are three times horizontally and twice vertically by the basis of CENTRE.

The convenient point of M machine is its constant magnification operation called scaling, not shearing. The technology that they wanted was scaling independently on the axis as shown in Figure 3.

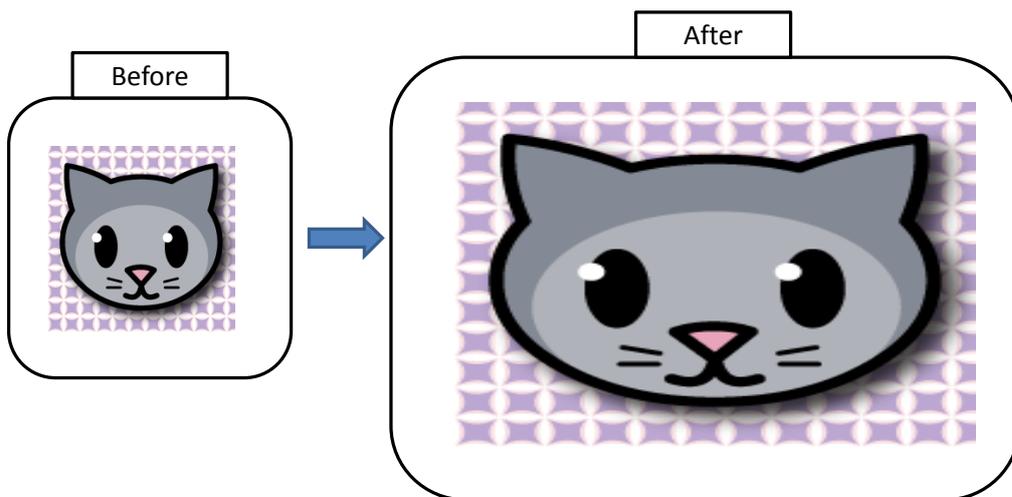


Figure3: The scaling transformation which is expressed by a matrix $\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$.

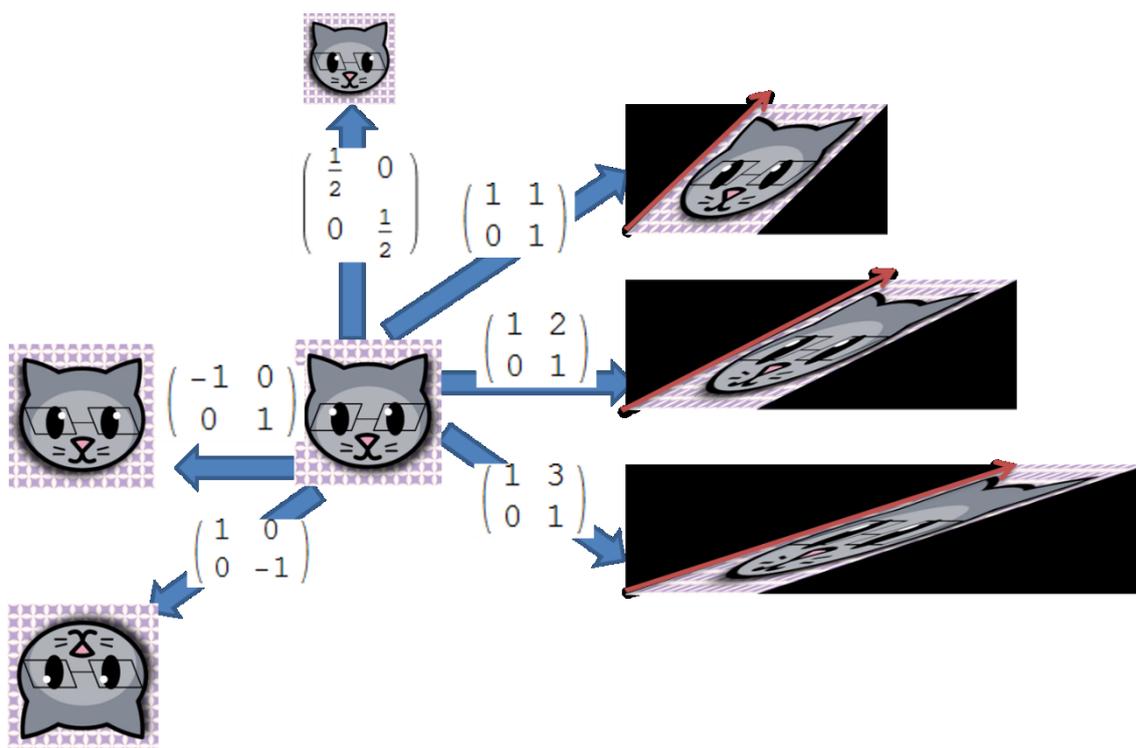


Figure4: The various transformation using the coordinate transformation matrix. Seeing the three transformations on the right side, we can see that the cat's right ear top (0, 1) gets expanded gradually.

Hearing this great news, Ms Diagonary rushed to his laboratory to watch the demonstration. In the demonstration, the input image is a face of Ms Diagonary (See Figure 3). Her face image is here scaled three times and twice orthogonally. Ms Diagonary felt proud to be a friend of Mr Orthom. At the same time, she got a bit cross to see her enlarged face. She delivered a monologue "I don't gain a weight."

This is because what they see is the same transformation. The same image transformation is expressed differently by different coordinate systems because their basis vectors are different. However, the enlargement factors are orthogonally three times and vertically twice and they are invariant even if we change the basis vectors.

Then Ms Diagonary tried to conduct this demonstration. But the transformation was not done well, sheared and rotated. She said "That was not a scaling. What is the trouble reason?"

After several minute thinking, Mr Orthom found the reason.

Because the basis vector of WEST country is different, the M machine did not work when Ms Diagonary input her coordinate-oriented image. Hearing the reason, Ms Diagonary asked him to change the basis vectors for her country WEST. Mr Orthom complied with her request. He said "We

can solve the problem taking the following steps.” He proposed the following transformation sequences:

- (1) Input the image expressed by the WEST coordinates.
- (2) Interpret the WEST coordinates to the CENTRE coordinates.
- (3) Magnify the image relative to the CENTRE basis.
- (4) Interpret the CENTRE coordinates to the WEST coordinates.
- (5) The image expressed by the WEST coordinates is output.

Let me explain the change of the bases.

First, let’s find the change-of-basis matrix P from the basis in CENTRE to the basis in WEST. In other words, let’s find the change-of-basis matrix P from the basis C to the basis W like the following:

$$C = \{(1,0), (0,1)\} \quad \text{and} \quad W = \{(1,0), (1,1)\}$$

Since C is the usual basis, write the vectors in WEST as columns to obtain the change-of-basis matrix:

$$P = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Although P is called the change-of-matrix from the original basis C to the new basis W , it is P^{-1} that transforms the coordinates of a vector relative to the original basis C into the coordinates of the vector relative to the new basis W . Inversely, it is P that transforms the coordinates of a vector relative to the new basis W into the coordinates of the vector relative to the original basis C .

By using this matrix P , let’s represent the point $(1, 1)$ in the WEST basis to one in the CENTRE basis. The answer is $(2,1)$ after the following calculation.

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Back to Figure 1, we can make sure that the location $(1, 1)$ in the WEST basis is the same as the location $(2, 1)$ in the CENTRE basis.

In conclusion, the right procedure of the magnification for the WEST country becomes as follows:

Input the image expressed by the WEST coordinates.

↓

P Transform from the WEST coordinates to the CENTRE coordinates.

↓

A Magnify the image.

↓

P⁻¹ Transform from the CENTRE coordinates to the WEST coordinates.

↓

The image expressed by the WEST coordinates is output.

Then find the inverse matrix of P, P^{-1} .

$$P = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad P^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

The scaling transformation is expressed as follows:

$$A = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$$

Then the following is the whole transformation matrix:

$$B = P^{-1} A P$$

This B is the matrix representation of the magnification relative to the basis WEST.

$$B = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}$$

Let's make sure that the input (1, 1) relative to the WEST basis will be magnified by three times

and twice: $\begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$

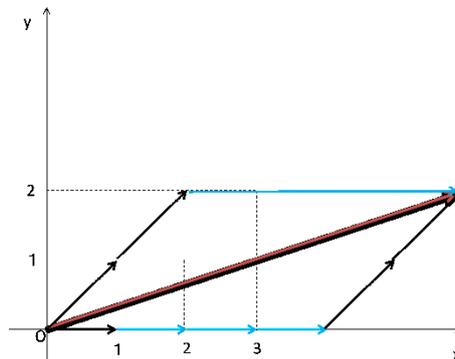


Figure5: The magnification (scaling) transformation of the WEST coordinates (1, 1) by the matrix B

$$\begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}.$$

Mr Orthom said to Ms Diagonary, “This matrix is the Magnification Machine for WEST country. In other words, this is the M Machine with the interpretation function for WEST country.” “Thank you so much, Mr Orthom.” She said to Mr Orthom. Then, she soon tried the magnification using this new version. This time, the transformation has been succeeded.

Now then, we will verify the correctness by the eigenvalues of the matrix B.

Do you remember how to find eigenvalues from a matrix?

The scalar λ is an eigenvalue which satisfies the characteristic equation $\det(B - \lambda I) = 0$.

The determinant of A is denoted by $\det(A)$ here. Solve the following equation

$$\det \begin{pmatrix} 3 - \lambda & 1 \\ 0 & 2 - \lambda \end{pmatrix} = 0$$

we get the eigenvalues $\lambda=3, 2$.

We found that the eigenvalues are not changed under the basis change.

In conclusion, the enlargement factors 3 and 2 of the M machine are corresponding to the eigenvalues of the magnifying transformation matrix.

Even if we add the interpretation function to the M Machine, namely under the basis change, the enlargement factors are invariant.

Let's go back to the $B=P^{-1} A P$

$$B = \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}$$

Concerning the magnifying matrix, let's calculate the eigenvectors. They are

$$\lambda=3 \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\lambda=2, \quad \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

We found that the eigenvectors are the WEST basis vectors.

Figuratively, let's consider the eigenvectors of the matrix corresponding to the M Machine with interpretation function. Then, the eigenvectors are to be each country's basis vectors.

The change-of-basis changes the matrix and the eigenvectors. But the eigenvalues are invariant.

Mr Orthom customized the M Machine for WEST. Then, for exports of the M Machine to other countries, customized interpretation functions have to be added. In other words, we have to find the change-of-basis matrix from CENTRE to the target country.

This is the end of the story.

I wonder if you could understand that the eigenvalues are invariant under the change of basis. Then every time you meet eigenvalues while you are reading math books, I hope that you remember this story and chuckle. Please enjoy various changes of bases.

The END