Extension of the "SNTV = d'Hondt" Theory and the Threshold of Exclusion to Multiple, Limited, Cumulative Vote *

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Abstract

Cox (1991) shows that, under the rationality assumption, SNTV and d'Hondt allocate the same seats to parties. This paper extends the theory to electoral systems of the multiple, limited and cumulative vote (MLCV) and gives plurality formula their corresponding PR systems which are equivalent to plurality formula. Moreover, it extends the threshold of exclusion to MLCV for more than one seat. Application to the Japanese cases, 1890-1898 and 1946, reveals how often undernomination, overnomination and failure of vote allocation occured in MLCV.

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INTRODUCTION

Cox (1991) shows that, under the rationality assumption, Single Non-Transferable Vote (SNTV) and d'Hondt allocate the same seats to parties. Cox (1990) illustrates that the number of votes each citizen can cast, the number of seats each district elects and whether voters can cumulate their votes affect position-taking incentives of parties and candidates. Combining both, this paper extends the "SNTV = d'Hondt" theory to electoral systems of the multiple, limited and (partial) cumulative vote (MLCV).

In the literature on the "SNTV = d'Hondt" theory, scholars have paid special attention to overnomination, undernomination and failure of vote allocation (Baker and Scheiner, 2007; Christensen and Johnson, 1995; Christensen, 2000; Cox and Niou, 1994; Cox and Rosenbluth, 1994; Kawato, 2004; Lijphart, Pintor, and Sone, 1986). If a party fields more (less) candidates than the number of seats derived by d'Hondt, given the number of votes it obtains, this is overnomination (undernomination). And if a party wins more (less) seats than the number of seats derived by d'Hondt, this is excess seats (insufficient seats). What if MLCV? How do we calculate the benchmark number of seats? What PR formula is equivalent to MLCV? So far, it has not been known. That's why the number of candidates and vote allocation are rarely studied except for SNTV.

MLCV was and is in use. Limited vote systems are and were used in British House of Commons (1867-85), Gibraltar's city council (-1969), Japan's lower house (1946), Kuwait (2008) and Spanish Senate (1977-) (Lijphart, Pintor, and Sone, 1986, 256). Cumulative vote was employed in English school board (1870-1902) (Newland, 1982, 84) and Illinois House (Cox, 1991, 120). But a more important reason why MLCV should be studied is that MLCV research leads us to theory on how elements of electoral systems work. For example, consideration of MLCV naturally enables us to extend the threshold of exclusion for more than one seat.

This paper is organized as follows. First, it introduces Head Start Divisior PR (HPR) formula and proves that it is equivalent to MLCV, adapting Cox (1991). The next section extends the threshold of exclusion to MLCV for more than one seat and demonstrates how elements of electoral systems ameliorate or deteriorate minorioty protection. Finally, application of HPR to the Japanese cases, 1890-1898 and 1946, reveals how often undernomination, overnomination and failure of vote allocation occured in MLCV.

THEORY

Setup

Suppose that a district will have a district magnitude of k, a number of parties equal to n, and a vote distribution v. In order to make it easy to compare this study with Cox (1991), this paper uses the same notation as Cox's (1991). The only exception is that, instead of vote share, this paper uses the number of votes, which are denoted by large letter V_i instead of small letter v_i which represents vote share (V_i/V where $V = \sum_i V_i$).

Plurality Formula. Cox (1991) considers SNTV. Top k vote getters are elected. This paper extends SNTV in the following way. Every voter has b votes and can cumulate c votes on a candidate. Votes remain to be non transferable. Note that $k \geq b \geq c \geq 1$. This paper calls this Multiple, Limited and Partial Cumulative Vote (MLCV). When $b = c = 1$, this is reduced to SNTV. When $k = b \neq 1$, this is not limited vote but bloc vote system. When $c = 1$, this is not cumulative. When $b = c \neq 1$, this is cumulative, not partial cumulative.

Proportional Representation (PR): Head Start Divisor. I shall denote the seat allocations made in this district by *a PR* algorithm as $s_1, s_2, \ldots, s_n$ (This subsection adapts Cox (1991, 122) and most of revised parts are *italicized* so that readers can notice key points easily). The set of all parties that win at least one seat will be denoted by W, while the set of all parties that win no seats will be denoted by L.

Let s-th divisor of a largest average method of PR be denoted by $D(s)$. For example, in the case of dHondt, $D(s) = s$; in the case of Saint Lague, $D(s) = 2s + 1$. This paper introduces the "head start divisor":

$$D(s) = \begin{cases} b/c & \text{if} \quad b/c > s \\ s & \text{otherwise.} \end{cases}$$

for positive integer s ($b/c \geq 1$). The PR with the head start divisior will be called HPR.

Lemma and Theorem

Suppose party h win at least one seat: $h \in W$. Then, for any largest average method of PR, the followiing lemma is established.

Lemma: $v_h/D(s_h) > v_j/D(s_j + 1)$

This says simply that the last winning average for a party that wins some seats must exceed the first losing average of any party (otherwise the latter party should have won another seat). Cox's Lemmas 1 and 2 (1991) are obtained when $s_j = 0$ (loser) and $s_j > 0$ (winner), respectively.

The *HPR* district will be compared to one operating under plurality formula with the same district magnitude, number of parties, and distribution of voter support among the parties. The plurality district, in other words, will differ only in that plurality formula rather than *HPR* is used.

The following theorem is proposed.

Theorem: For a given district magnitude (k), distribution of vote support (v), and expected distribution of vote support (e), *a number of ballots every voter has (b), a number of ballots every voter cumulate on a candidate (c)*, the allocation of seats under *HPR* and the equilibrium allocation of seats under $MLCV$ are identical, if (1) $e = v$; (2) *party* i under plurality can allocate their total vote equally among their nominees *when* $D(s_i)$ *is an integer; otherwise, party* i *can allocate* $V_i/D(s_i)$ *votes to each of* $\lfloor D(s_i) \rfloor$ *nominees and* $(1 - (\lfloor D(s_i) \rfloor / D(s_i)))V_i$ *to a nominee (thus, all parties field* $\lceil D(s_i) \rceil$ *candidates);* (3) all parties seek to maximize the expected number of seats that they will win; and *(4) voters cast all* b *votes they have. (No (partial) abstention).*

$\lceil \cdot \rceil$ is ceiling, while $\lfloor \cdot \rfloor$ is floor. The fourth condition is added by this paper to Cox (1991). Under this assumption, the number of voters who vote for party i is calculated as $W_i = V_i/b$. Thus, the maximum a party i's candidate has is cV_i/b. If multiple candidates have the same number of votes but k is not sufficient to elect them all, some of them are chosen by lottery.

Proof

This part adapts Cox (1991, 123-4). When the number of candidates and seats can be both single and multiple, this paper write only plural expression for simplicity of presentation.

What I will show first is that, if each party in the plurality district nominates a specified number of candidates and spreads its total support among them, *in principle,* in an evenhanded way, then each will win exactly the same number of seats as it does under *HPR*(*identity*). I will then show that, if parties expectations about the distribution of voting support are accurate and each party seeks to maximize the expected number of seats that it will win, then no party can unilaterally improve on the specified situation (i.e., it is an equilibrium) (*equilibrium*). Finally, I will show that there is no other equilibrium that yields a different allocation of seats (*uniqueness*).

Identity. Suppose, then, that each party in the plurality district nominates as follows. If it would win no seats under *HPR*, then it nominates $\lceil D(s_i + 1) \rceil = \lceil b/c \rceil$ *candidates (where* $s_i = 0$*) and give* $V_i/D(s_i + 1) = bW_i/(b/c) = cW_i$ *votes to* $\lfloor D(s_i + 1) \rfloor = \lfloor b/c \rfloor$ *candidates* *and, if any* $(D(s_i+1) > \lfloor D(s_i+1)\rfloor)$*, the rest* $(1-(\lfloor D(s_i+1)\rfloor/D(s_i+1)))V_i$ *votes to one remaining candidate.* Otherwise, it nominates a number of candidates equal to the number of seats that it would win under *HPR* and spreads its total vote support equally among these nominees. In terms of the notation established above, if $i \in L$ then i nominates $\lceil b/c \rceil$ *candidates and these candidates receive* a vote share of v_i; if $i \in W$ then i nominates s_i candidates and each receives a vote share of v_i/s_i.

Given these nomination and vote allocation strategies, will each party get the same seat shares as under *HPR*? The answer is 'yes' and can be seen as follows. Consider first those parties that would win no seats under *HPR* (the *HPR* losers). If one of them, say j, wins a seat under plurality then its candidate must finish in the top k. Because there are k nominees from parties in W, however, this means that party j's nominee must beat at least one nominee from a party in W, say party h. But this implies that $V_j/D(s_j+1) > V_h/D(s_h)$, which contradicts *Lemma.* So no *HPR* loser can win under plurality, if parties nominate and allocate their votes in the fashion specified. Because the *HPR* winners (parties that would have won at least one seat under *HPR*) nominate all together exactly k candidates, the fact that no *HPR* loser can win a seat means that each succeeds in electing all its candidates under plurality formula. Hence, the allocation of seats under *HPR* and plurality is identical, if parties under plurality nominate and allocate their votes in the fashion posited.

Equilibrium. The next question is whether there is any reason to expect that parties under plurality will in fact nominate and allocate their votes as posited. One aspect of this question concerns whether the posited situation (in which each *HPR* loser nominates $\lceil D(s_i+1)\rceil = \lceil b/c \rceil$ *candidates*, each *HPR* winner nominates the same number of candidates as it would win under *HPR*, and all *winners* spread their total vote equally among their nominees) is an equilibrium.

The definition of an equilibrium, roughly, is a situation in which no party can change the number of its nominees (or the vote shares they get) in such a fashion as to gain an extra seat, if other parties maintain their current strategies. What needs to be added to this definition to make it more precise is simply a more explicit recognition that parties must make their nomination and vote allocation decisions before actually observing their total vote. That is, party i does not observe the vote distribution v before it decides on its nominations and vote allocation. All it has is an expectation, e of what this vote distribution will be like. I shall assume to begin with that each party has the same expectation and that this expectation is correct, that is, $e = v$.

Under this assumption, the posited situation is in fact an equilibrium. No *HPR* loser can do anything to gain an extra seat; *to nominate more candidates* would just spread an already insufficient vote total more thinly. *Nomination of less candidates would not increase the votes any candidate obtain because candidates have already earned the maximum number of votes which all W_i party i supporters can cumulate on one candidate, cW_i.* Nor can any *HPR* winner gain an extra seat. Nominating fewer candidates obviously cannot win more seats. Nominating more candidates is equally ineffective. Suppose some party $j \in W$ nominates one more candidate and distributes its total vote in some fashion among its $s_j + 1$ nominees. The only way for it to win an extra seat is if all of its nominees win seats, which requires that each of them, even the weakest, beat all the candidates from some other *HPR* winner. Because the weakest nominee from party j can get at most $v_j/(s_j + 1)$, this means that there must be some $h \in W$ such that $v_j/(s_j + 1) > v_h/s_h$. But this violates *Lemma*. Nominating even more candidates only spreads the vote thinner. Hence, the posited situation is in fact an equilibrium, if expectations are accurate.

Uniqueness. The final question is whether there are equilibria other than the one just identified. In general, the answer is 'yes'. There are many other equilibria. But none of them yields a different allocation of seats.

These assertions can be illustrated by referring to Table 1 *in Cox (1991)* where Party 1, 2, 3, 4 and 5 obtains 36, 30, 14, 12 and 8 votes, respectively, with district magnitude of five. In this case, the suggested equilibrium under plurality formula is for parties 1 and 2 to nominate two candidates apiece (dividing their total vote equally among them), while all the other parties nominate one candidate each. This would produce two candidates from party 1 with 18 votes each, two from party 2 with 15 votes apiece, one from party 3 with 14 votes, and so forth. If party 1 were to allocate its vote differently, say giving 20 to its first nominee and 16 to its second, nothing of importance would change. That is, the resulting new situation would still be an equilibrium and the seat allocations it would entail would be no different. As can be readily seen, there are many other similarly inconsequential changes that parties 1 and 2 can make in the distribution of their vote. Thus, there are many equilibria.

No other equilibrium yields a different allocation of seats, however. Consider any set of nomination and vote allocation decisions that leads to a seat allocation, say, $s^* = (s_1^*, \ldots, s_n^*)$, different from the original equilibrium allocation, $s = (s_1, \ldots, s_n)$. Because $\sum s_j^* = \sum s_j = k$, the situation is zero-sum. If s^* differs from s as posited, then, there must be some party, say h, that loses seats ($s_h^* < s_h$) and some other party, say j, that gains ($s_j^* > s_j$). Consider first the case in which j did not win any seats in the original equilibrium ($s_j = 0$). Because $s_j^* \geq s_j+1$, $v_j/D(s_j + 1) \geq v_j/D(s_j^*)$. If s^* is an equilibrium allocation of seats, then *Lemma* implies that $v_j/D(s_j^*) > v_h/D(s_h^*+1)$. Because $s_h^*+1 \leq s_h$, it follows that $v_h/D(s_h^*+1) \geq v_h/D(s_h)$. Thus, if s^* is an equilibrium, then $v_j/D(s_j + 1) > v_h/D(s_h)$. But this violates *Lemma since* $s_h > 0$. Hence, s^* cannot be an equilibrium. A similar argument works for the case in which $s_j > 0$.

Relaxing Some Assumptions

The condition (4) can be relaxed. Then, partial abstention (plump) is possible, though rational voters will never abstain. It does not increase the number of seats they obtain.

If $b > k$, the argument above still holds. Even in the case of $b/c > k$, the theorem is true, even though parties field more candidates than the number of district size.

Cox (1991, 124-5) shows that, even if some assumptions are relaxed, his theorem is the case. Similar arguement applied to the theorem of this paper.

Extended Threshold of Exclusion

HRP expression of MLCV suggests how much MLVC is in favor of minority groups, depending on parameter values. One way to measure the degree is the threshold of exclusion. It is the minimm vote share $\bar{v}$ to guarantee (or the sufficient condition of) a seat for party i even if the situation is worst, that is, there is only one alternative party j to which all the other votes $(1-\bar{v})V$ are cast.

When, $b = c = 1$, $\bar{v} = 1/(1+k)$. Why? Suppose otherwise. This implies that at least k winners of party j obtain larger than $\bar{v}$. The total of vote share party i and j is more than $\bar{v}(k+1) = 1$, a contradiction. This is the usual proof, though HRP expression of MLCV gives another appearance of proof.

Suppose that party i follows the nomination and vote allocation strategy of Theorem and obtain s_i seats. According to Theorem, party j's best response is also the nomination and vote allocation strategy of Theorem and obtain $s_j = k - s_i$ seats. By Lemma, it follows that

$$v_i/D(s_i) > v_j/D(s_j+1)$$

$$v_i/D(s_i) > (1-v_i)/D(k-s_i+1)$$

$$v_i > \bar{v}(s_i) \equiv D(s_i)/(D(s_i)+D(k-s_i+1))$$

Hence, if $v_i > \bar{v}(s_i)$, party i is guaranteed at least s_i seats whatever strategy party j follows. QED.

When $b = c = 1$, $\bar{v}(1) = 1/(1+k)$, namely, the threshold of exclusion. When $b > 1$ and $c = 1$, $\bar{v}(1) = b/(b+k)$ (Newland, 1982). Bloc vote raises the hurdle for minority. But the second proof described above naturally leads us to the following questions. What if (partial) cumulation is allowed ($b \geq c > 1$)? How about the minimum vote share to guarantee *not one but* s seats for party i even if the situation is least favorable? The answer is the *extended* threshold of exclusion, $\bar{v}(s)$.

This presentation well reveals how much these factors of electoral system affect minority protection. If $b = c$, $\bar{v}(1) = 1/(1+k)$, which is reduced to the (non-extended) threshold of exclusion. Thus, cumulative vote dilutes bloc votes. It is well known that cumulative vote usually improves political representation of minority population (Bowler, Brockington, and Donovan, 2001; Cooper, 2007). But this conjecture is not necessarily well established, especially by way of formal theory.

In general, consider $\bar{v}(1) = \frac{b/c}{(b/c)+k}$.

$$\frac{\partial \bar{v}(1)}{\partial k} < 0$$
$$\frac{\partial \bar{v}(1)}{\partial b} > 0$$
$$\frac{\partial \bar{v}(1)}{\partial c} < 0$$

District magnitude k and (partial) cumulative vote (c) ameliorate minoroty representation, while (limited) bloc vote (b) deteriorate it.

This implication can be generalized into the case of $s > 1$. Note

$$\bar{v}(s) = \begin{cases} \frac{b/c}{(b/c)+k} & \text{if} \quad b/c > s \\ \frac{s}{s+k} & \text{otherwise.} \end{cases}$$

It follows

$$\frac{\partial \bar{v}(s)}{k} < 0$$
$$\frac{\partial \bar{v}(s)}{b} \geq 0$$
$$\frac{\partial \bar{v}(s)}{c} \leq 0$$
$$\frac{\partial \bar{v}(s)}{s} \geq 0$$

Figure 1 illustrates the cases of $k = 5$ with $(s, c) = (1, 1), (2, 1), (1, 2), (2, 2)$.

The extended threshold of exclusion can be applied to aother type of plurality formula than MLCV, if it has its equivalent formula of largest average PR. If its divisor is $D(s)$, the extended threshold of exclusion is $\bar{v}(s) \equiv D(s)/(D(s) + D(k - s + 1))$. In general, it follows that $\bar{v}(s) = 1 - \bar{v}(k - s + 1)$. This implies that, if a party is guaranteed s seats, the other parties cannot have $k - s + 1$ seats in sum. Though this seems to be a matter of course, it has not been made explicit.

Japan, 1890-1898 and 1946

This paper provides MLCV with the number of seats derived by HPR. This section applies HPR to the Japanese general elections in 1890-1898 and 1946 when multiple votes are employed (only in 1946, limited vote was used).[1]

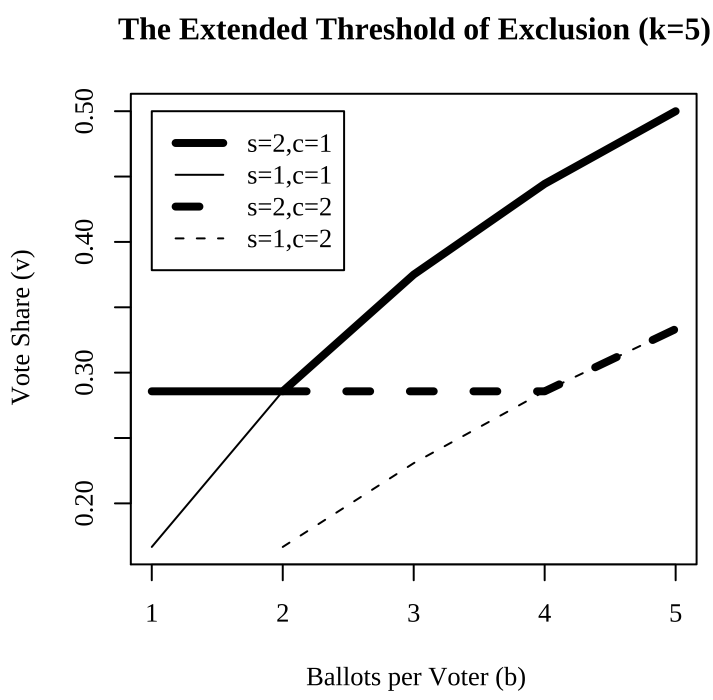
Table 1 reports the percentage of combinations of overnomination, undernomination, excess seats and insufficient seats for the two largest partie sin seats (for every election, the first and second rows present the largest and next parties, respectively). In the 1890s, when 43 districts have two seats with two ballots for every voter (bloc vote, no cumulation) and 214 districts have one seat, parties rarely lose seats compared with HPR seats. In 1946,

[1]Data is based on Kawato and Kawato (1997).

when 53 districts have 4 to 14 seats with two or three ballots for every voter, most parties committed overnomination and, in one third of districts, suffer from insufficient seats.

Seat Candidate	± 0 ± 0	± 0 $+$	$+$ $+$	$-$ $+$	$-$ ± 0	$-$ $-$	Number of Districts
1890	69.4	2.7	27.0	0.0	0.0	0.9	111
	69.6	3.8	24.1	0.0	1.3	1.3	79
1892	22.3	68.1	2.7	3.7	2.7	0.5	188
	80.0	12.5	7.5	0.0	0.0	0.0	80
1894(Mar.)	39.2	52.4	2.1	1.6	3.2	1.6	189
	28.0	66.4	4.7	0.0	0.9	0.0	107
1894(Sep.)	34.7	59.6	2.1	1.0	2.6	0.0	193
	26.9	69.7	2.5	0.0	0.8	0.0	119
1898(Mar.)	32.9	57.2	5.2	0.6	3.5	0.6	173
	51.9	39.7	3.8	1.5	3.1	0.0	131
1898(Aug.)	53.7	33.3	4.8	1.4	5.4	1.4	147
	48.7	38.0	8.0	0.7	2.0	2.7	150
1946	0.0	37.7	28.3	34.0	0.0	0.0	53
	0.0	40.4	21.2	38.5	0.0	0.0	52

Table 1: Percentages of Districts for Seat-Candidate Categories (Columns) for the Largest Party (1st Rows) and the Second Party (2nd Rows) for MLCV Elections in Japan, 1890-1898 and 1946


The Extended Threshold of Exclusion (k=5)
s=2,c=1
s=1,c=1
s=2,c=2
s=1,c=2
0.20
0.30
0.40
0.50
Vote Share (v)
1
2
3
4
5
Ballots per Voter (b)

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